# THE TREATMENT OF BRANCH CUTS IN ADAPTIVE OPTICS.

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## 1 ABSTRACT

Branch cuts are discontinuities in the wavefront phase which occur in correspondence with the presence of zeros in an intensity distribution. Most adaptive optics wavefront sensing techniques do not detect such discontinuities, and most wavefront modulators are unable to correct them. We have therefore studied the formation and potential correction of branch cuts. This paper presents a simplified example and examines in detail the form of branch cuts and the effects correction may have on the reconstructed wavefront.

## 2 INTRODUCTION

The presence of discontinuities in the phase function associated with wave trains was first discussed in a celebrated paper by Nye and Berry [1]. These discontinuities (branch cuts) are associated with zeros in the field amplitude. In fields such as optics, wavefunctions expressed as complex functions of one complex variable are causal (band limited) and are therefore analytic functions [2]. The logarithm of such a function is also an analytic function, except at those points where the modulus of the field goes to zero, at which there are logarithmic singularities (these are branch points). In two dimensions the wavefunction is a complex function of two complex variables and the branch points are connected by discontinuities in the phase of the function (branch cuts). The logarithm of the function is multi-valued and the function lies on Reimann sheets separated by $2\pi$ in phase. The branch cuts represent a tear in the Reimann sheet that connect two sheets together.

The wavefunction phase on a closed circuit around an isolated branch point in the real plane increases (or decreases) by $2\pi$. Two branch points of opposite handedness may be connected by a branch cut so that a circuit passing between the two branch points shows a $2\pi$ phase change whilst a circuit enclosing both branch points shows no discontinuity in phase. In problems of optical propagation through stochastic media the phase errors induced by the transmission medium can lead to the formation of scintillation in the form of caustics or speckle. The density of branch points in speckle patterns has been investigated [3] showing that the density of branch points increases non-linearly as the turbulence strength increases and the speckle pattern approaches a fully-developed Gaussian speckle pattern.

In adaptive optics one need to be concerned both with the ability to detect such phase discontinuities and whether or not one should correct them if detected. Previous examination of this question [4, 5] has shown unexpected results. However, previous work has examined structurally complex fields and tended to concentrate on regimes where severe turbulence leads to a significant density of zeros and/or to approximations of the functions describing the wavefields involved.

Here we have attempted to use a numerical experiments based on a simple mathematical description in which we can control the density and location of the branch points in the real plane for a band-limited complex function of two complex variables. By this means we seek to elucidate the structure of branch cuts in bandlimited functions. In particular, if branch cuts are an essential feature associated with the existence of branch points, is desirable to 'correct' the branch cuts or does such correction inevitably lead to the function becoming non band-limited?

## 3 METHODOLOGY

Following van Toorn et al [6], we start with an analytic description of the Fourier transform of a plane wave that has uniform amplitude within a disc and is identically zero outside the disc. The zeros of this simple function, that is commonly-encountered in optics, form concentric rings in the real plane. It is interesting to note at this point that the discontinuities in phase associated with these zeros are of height $\pi$ and not of height $2\pi$. Starting from this simple band-limited function we displace the zeros from the real plane in such a way that the position of the branch points are known. This procedure permits us to produce complex functions with a known number of branch points in known locations, whilst preserving the zero density and thus the bandlimited nature of the function.

Before proceeding to this analysis a few other comments about the nature of the complex functions thus generated is in order.

The two dimensional Fourier transform can be factorised as a series of one-dimensional Fourier transforms. Thus

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, f(\xi,\eta)\exp\left(-i[x.\xi + y.\eta]\right) \\ &= \int_{-\infty}^{\infty} d\xi\, \tilde{f}(\xi,y)\exp\left(-ix.\xi\right) \end{aligned} \tag{1}$$

where $f(\xi,\eta) = 1 \forall \xi^2 + \eta^2 \le \rho; f(\xi,\eta) = 0 \forall \xi^2 + \eta^2 > \rho$, $\rho$ being the radius of the disc and $\tilde{f}(\xi,y)$ the Fourier transform w.r.t. $\eta$ evaluated for a given $\xi$. Note that $\tilde{f}(\xi,y) = 0 \forall |\xi| > \rho$. Note also that in general $\tilde{f}(\xi,y) \to 0$ quadratically as $|\xi| \to \rho$. Thus along any section $y$ we have that, in general, $F(x,y)$ is an entire function of exponential type with zeros having an asymptotic distribution with spacing $\pi/\rho$ and lying on the real axis. In consequence we expect that functions with a convex support will have concentric contours of real zeros asymptotically. Where the convex support is a disc, the zeros will be asymptotically distributed as uniformly-spaced rings. The interaction of branch cuts with these rings of zeros is a point requiring investigation.

The Fourier transform of the function $f(\xi,\eta)$ defined above is a first order Bessel function divided by its argument, thus

$$F(x,y) = \frac{J_1(r)}{r} \quad \text{where } r^2 = x^2 + y^2. \tag{2}$$

This function, being quasi-one-dimensional, can be factorised thus:

$$F(x,y) = \prod_n \left(1 - \frac{r^2}{j_n^2}\right), \tag{3}$$

where $j_n$ is the (real valued) radius of the nth ring of zero value and the sum is taken to infinity.

If we wish to change the position of one or more of the zero-value rings, we can do this by defining the function

$$p_k(x,y) = 1 - \left[\frac{(x+a_k)^2 + (y+b_k)^2}{j_k^2}\right] \tag{4}$$

which can be substituted into the power series expression for instead of the appropriate factor when $j_k = j_n$ in the infinite product. Note that $p_k(x,y)$ has real zeros when

$$x\left(x + 2\,\mathrm{Re}\{a_k\}^2\right) - \mathrm{Im}\{a_k\}^2 + y\left(y + 2\,\mathrm{Re}\{b_k\}^2\right) - \mathrm{Im}\{b_k\}^2 - j_k^2 = 0 \tag{5}$$

and

$$\left(\mathrm{Re}\{a_k\} + x\right)\mathrm{Im}\{a_k\} + \left(\mathrm{Re}\{b_k\} + y\right)\mathrm{Im}\{b_k\} = 0. \tag{6}$$

where $\mathrm{Re}\{\ \}$ and $\mathrm{Im}\{\ \}$ means real and imaginary part of the value in braces.

Defining

$$h(x,y) = \prod_{\substack{n \\ n \neq k}} \left[ 1 - \frac{r^2}{j_n^2} \right] \tag{7}$$

we can generate a new function

$$F'(x,y) = h(x,y) \times p_k(x,y)\,. \tag{8}$$

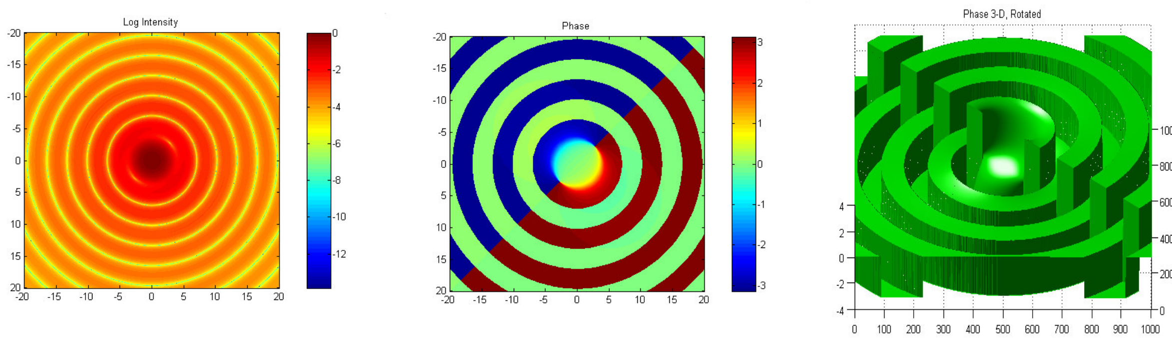
If $p_k(x,y)$ is an analytic function the new function $F'(x,y)$ will have the same band limit as our original function $F(x,y)$. For computational convenience, rather than evaluate the infinite series represented in equation (7) we choose to construct $F'(x,y)$ from

$$F'(x,y) = \frac{2J_1(r)\, p_k(x,y)}{r\left(1 - \dfrac{r^2}{j_k^2}\right)}\,. \tag{9}$$

This formulation is numerically stable provided that the sampled $(x,y)$ values do not lie exactly on one of the zeros of the initial function or on the origin. Since the function is defined analytically it can be computed easily and at very high resolution over localised regions of the $(x,y)$-plane.

## 4 SIMULATIONS

Our initial choices for $a_k, b_k$ was to set these to constant, imaginary values for one or more of the first few zeros of $J_1(r)$. With $a_1 = b_1 = -0.5i$, inspection of equation (6) shows that the zero must lie along a line drawn at $45^{\circ}$ through the origin in $(x,y)$ space. Examination of equation (5) shows that there are two zeros symmetrically placed with respect to the origin and slightly further from the origin than the original ring of zeros. (Note: for reasons not understood, the software package used to generate the figures has rotated some of the phasemaps through $90^{0}$.)




(a) (b) (c)

Fig. 1. Images of function with one pair of added branch points. $a_1 = b_1 = -0.5i$.
(a) Log Intensity, (b) Phase, (c) Phase 3-D.

The first feature to note from the phase map is that each of the zeros on the annulus with radius $j_1 = 3.8317$ is a branch point, as expected. The phase evaluated along a circuit enclosing the zero climbs through $2\pi$ and returns discontinuously to its starting value. The second feature to note is that along a diagonal line through the branch points there are a series of disconnected discontinuities in the phase function. The phase function consists of a series of concentric annuli, alternate annuli have a 2π discontinuity from –π to π in the phase function along the diagonal through the branch points and oriented along that diagonal. The intermediate annuli have no discontinuity and have a phase value ~0. There are discontinuities of ±π between alternate annuli beyond the first. Note that the discontinuities in the outer annuli are not branch cuts, as may be illustrated by changing the phase of the function by a small amount (0.08 π), leading to the phase diagram below:

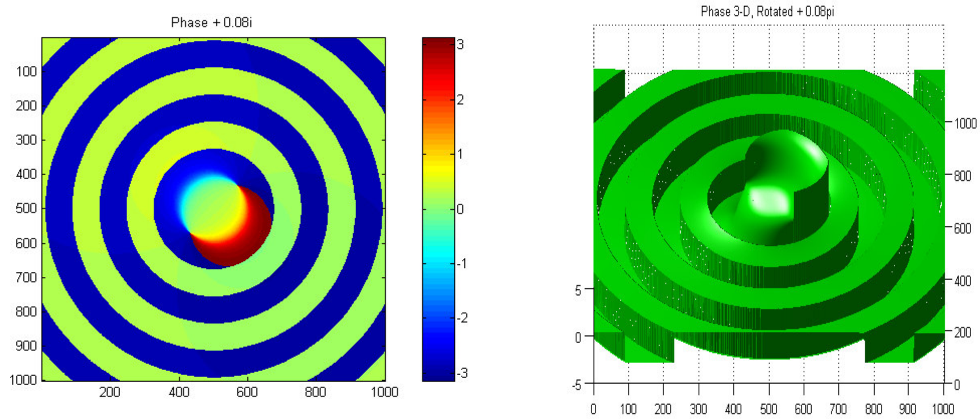



Fig. 2. Phasemap for one pair added branch points + 0.08π added constant phase
(a) Phase, (b) Phase 3-D

In terms of what one might generally anticipate these results seem to be unexpected. There are two branch points and a 2π discontinuity in the phase function on the inner annulus, but there is no branch cut joining the branch points. Instead the branch cuts end on the next circle of zeros without the formation of a branch point and a π discontinuity in the phase runs between the branch points along the circle of zeros. Thus the branch cuts each have a branch point at one end only. As the values $a_1$ and $b_1$ decrease the positions at which the two branch cuts meet the zero ring move towards each other, touching at a value slightly less than $a_1 = b_1 = -0.44i$. At this point the branch cut links the two branch points and as the values tend toward zero the branch cut tends to lie closer to the position occupied by the zero ring when $a_1 = b_1 = 0$ (See Fig. 3., below).

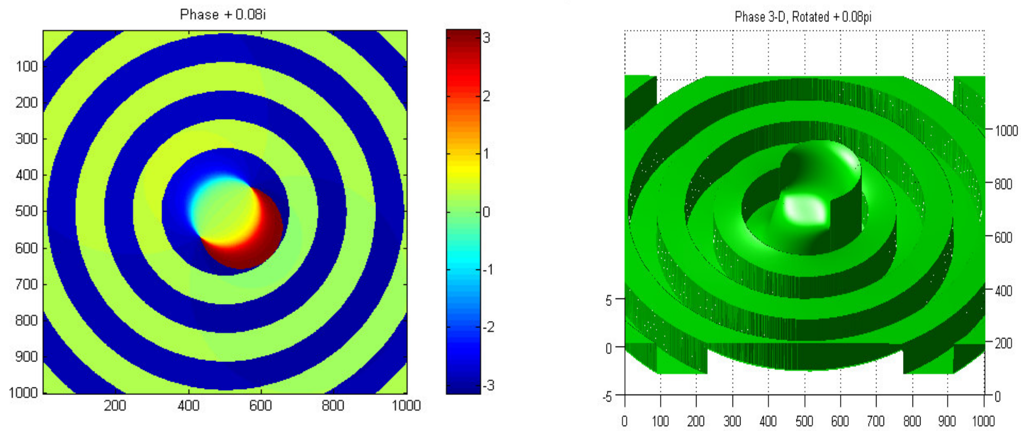



Fig.3. Phasemaps for $a_1 = b_1 = -0.44i$. Added constant phase = 0.08π.

A similar result can be achieved with the addition of further small phase changes ($a_1$ and $b_1$ remain unchanged, with values of –0.5i). From Fig. 4. it can be seen that for phase shifts greater than or equal to 0.09π, the 2π discontinuity is now fully contained between the first and second zero ring, and hence a single branch cut joins the two oppositely handed branch points.

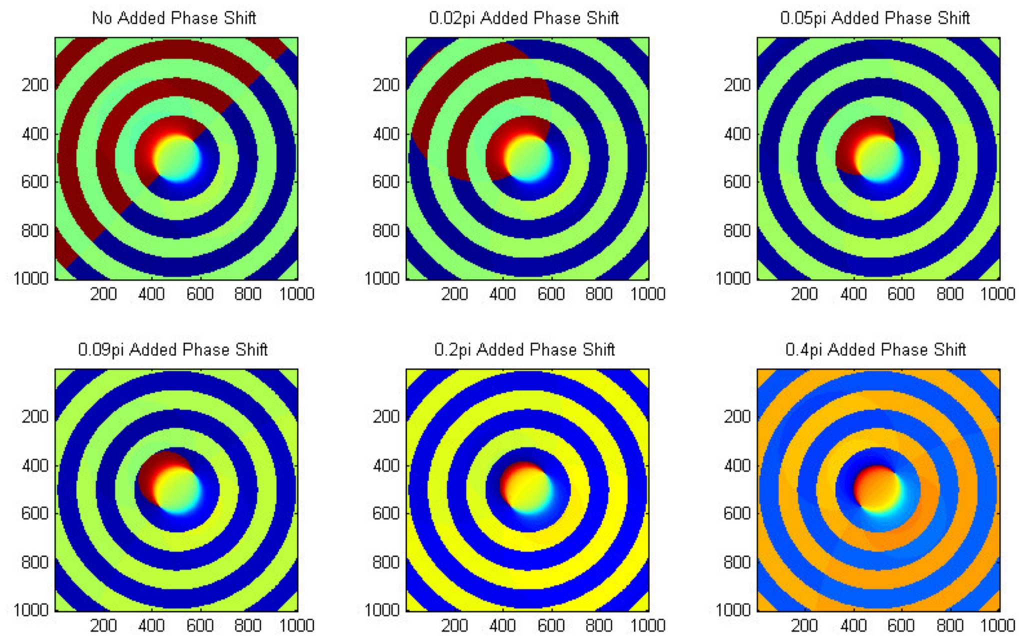
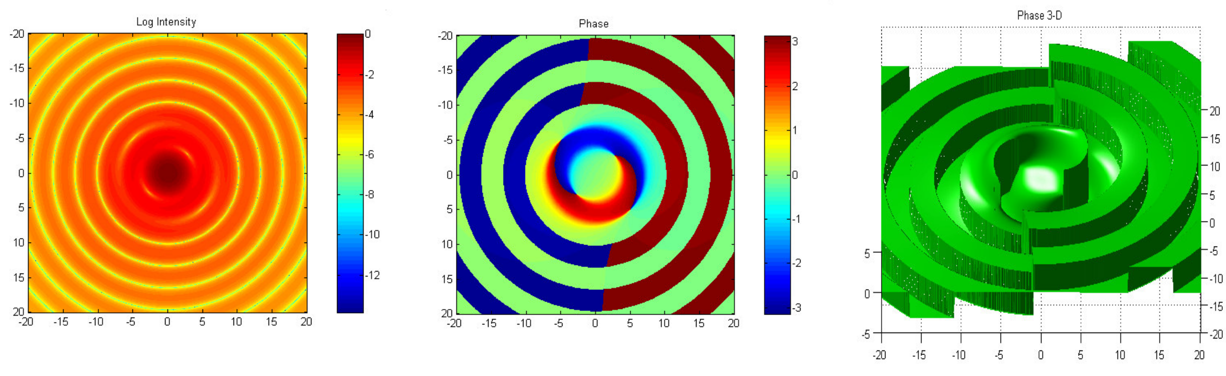



Fig. 4. Phasemaps for $a_1 = b_1 = -0.5i$, increasing amounts of constant phase added.

Adding a constant phase increment causes the area bounded by the imaginary zero contour line defining the discontinuities to reduce. This eventually leads to the case where the branch cut lies along the zero ring on which the branch points lie, and thus it lies in a region of low intensity.

If we now add two further branch points by removing the second circle of zeros and placing isolated zeros on the opposite diagonal as the first two ($a_2 = -b_2 = -0.5i$), we obtain two branch cuts running around the figure as shown below, each connecting two branch points belonging to the opposite zero contours.




(a) (b) (c)

Fig. 5. Images of function with two pairs of added branch points. $a_1 = b_1 = -0.5i$, $a_2 = -b_2 = -0.5i$.
(a) Log Intensity, (b) Phase, (c) Phase 3-D.

Again, the discontinuities in the outer annuli are not branch cuts. In this case the branch cut spirals across the annulus between the original first and second rings of zeros and have a branch point at each end. These branch cuts have the usual 2π discontinuity. If we examine the effect of adding a phase shift to this function, it can be seen that the branch cuts running between the annuli eventually split and, as in the previous example, the branch cuts run between pairs of branch points on the same zero ring.

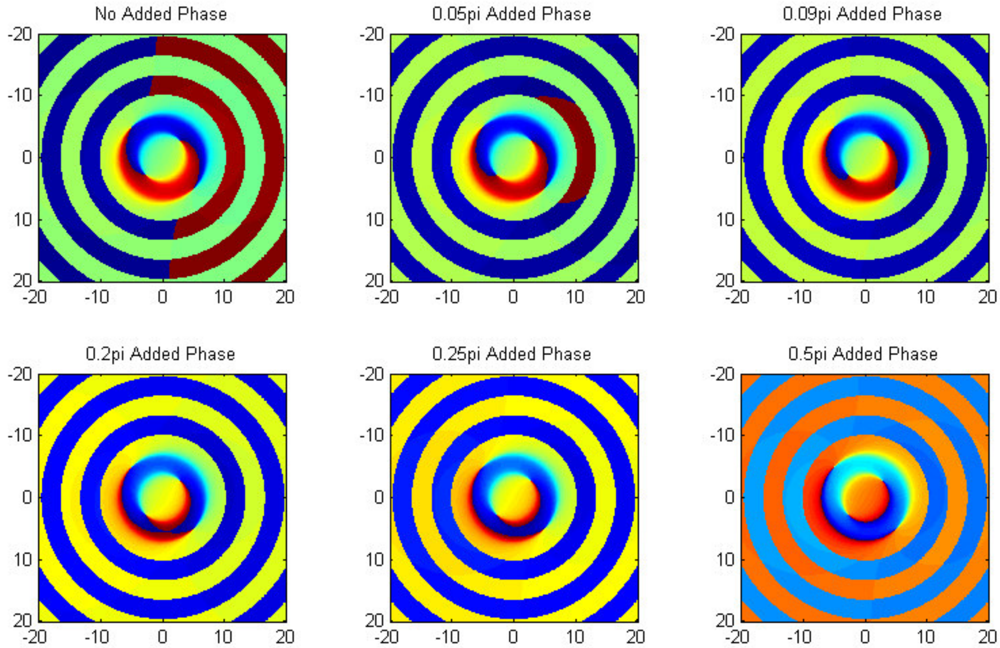



Fig. 6. Phasemaps for two pairs of added branch points. $a_1 = b_1 = -0.5i$, $a_2 = -b_2 = -0.5i$, increasing amounts of constant phase added.

Adding a further pair of zeros on the third zero ring, using $a_3 = -0.2i, b_3 = 0.25i$, and again adding constant phase to the function leads to similar results, with all three pairs of branch points tending to be connected by branch cuts along the radius of the annulus on which they lie (see Fig. 7.).

On the basis of these few examples we conclude that it seems unlikely that a branch cut will ever cross a line of real zeros. Since the asymptotic behaviour of realistic band-limited functions will have concentric real zero contours in the limit, we conclude that the branch points of such functions cannot be connect to infinity but must be connected pairwise within the region close to the optical axis. However, this connection can lie along a ring of zeros in which case the discontinuity connecting the branch points can be of height π and not 2π.

A further conclusion would appear to suggest that the simple process of adding a constant phase to the function can be used to re-map the locations of the branch cuts, and position them such that they lie in regions of low intensity only. Where this can be done, it would appear that branch cuts should not have a significant impact on phase reconstruction.

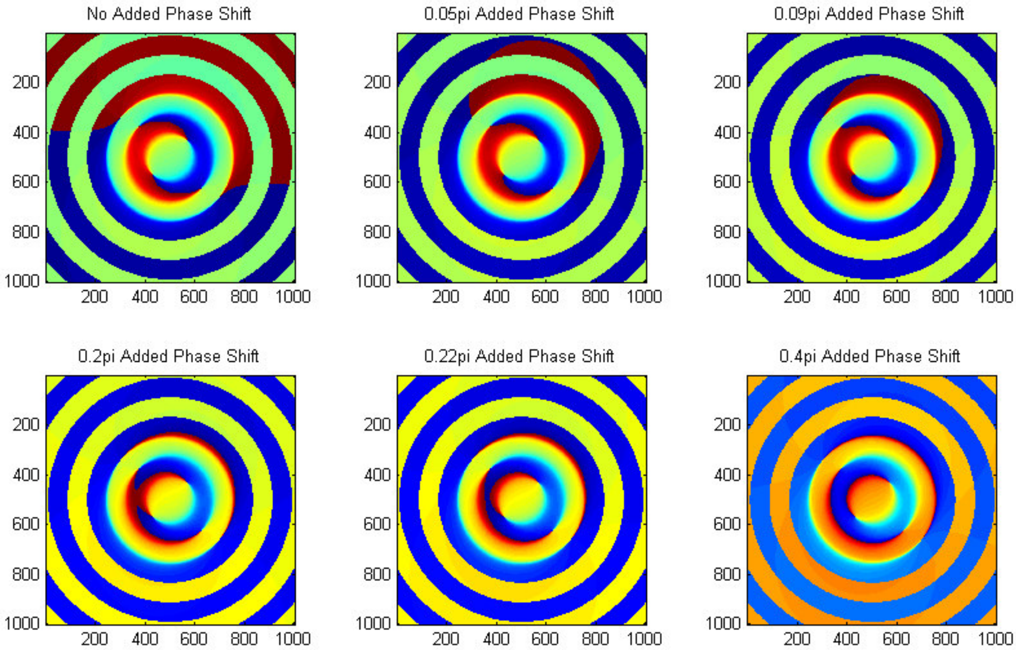



Fig. 7. Phasemaps for three pairs of added branch points. $a_1 = b_1 = -0.5i$, $a_2 = -b_2 = -0.5i$, $a_3 = -0.2i$, $b_3 = 0.25i$, increasing amounts of constant phase added.

In order to investigate more complex behaviour we have also investigated cases where the $a_k$ and $b_k$ in equation (4) can be functions of $(x, y)$. In the examples presented here we have chosen the functional relationships

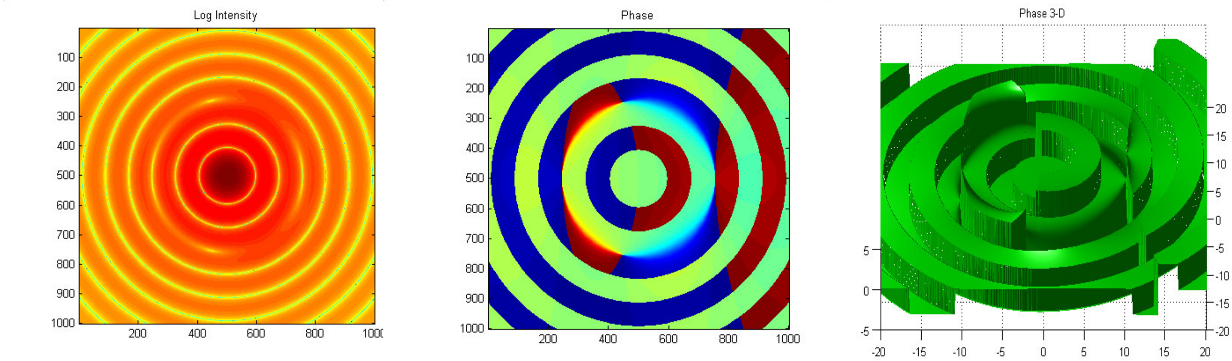
$$a_k(x, y) = i\cos\left(\frac{\pi x}{2 j_k}\right); b_k(x, y) = i\sin\left(\frac{y}{50}\right) \tag{10}$$

From equation (6) real zeros can occur only when

$$x\cos\left(\frac{\pi x}{2 j_k}\right) + y\sin\left(\frac{y}{50}\right) = 0 \tag{11}$$

Thus real zeros can occur for $x = nj_k$, and $y = 25m\pi$, $n, m$ integer. For $y = 0$ and $x = j_k$ there will be zeros at $x = \pm j_k$.

The situation where $j_k = j_3 = 10.174$ was modeled. Fig. 8 shows the intensity and phase of the resulting function when $a_k$ and $b_k$ are included into $p_k(x, y)$ (see Equation (4)). It can be seen that there are four branch points generated, and all are lying on the $j_3$ ring. Branch cuts appear to join up three of these points, however, the far right branch point appears to be isolated, with branch cuts running away from it as before, through alternate zero rings and between π phase steps.




(a) (b) (c)

Fig. 8. Images of function with functions incorporated for $a_3$ and $b_3$.
(a) Log Intensity, (b) Phase, (c) Phase 3-D.

As with the previous function, constant phase values were added to this phasemap, and the results plotted in Fig. 9. below. It can be seen that the three left-hand branch points behave as before, with the connecting branch cut tending towards the annulus on which all the branchpoints lie. However, the branchpoint on the right hand side remains unconnected. Indeed, as the phase value is increased the discontinuity running from the branch point seem to disappear altogether. Plotting the positions of the zero contour lines shows that the branch point is always connected to an imaginary zero contour, but as the added phase is increased, the contour it is associated with changes to the same one as the other three branchpoints.

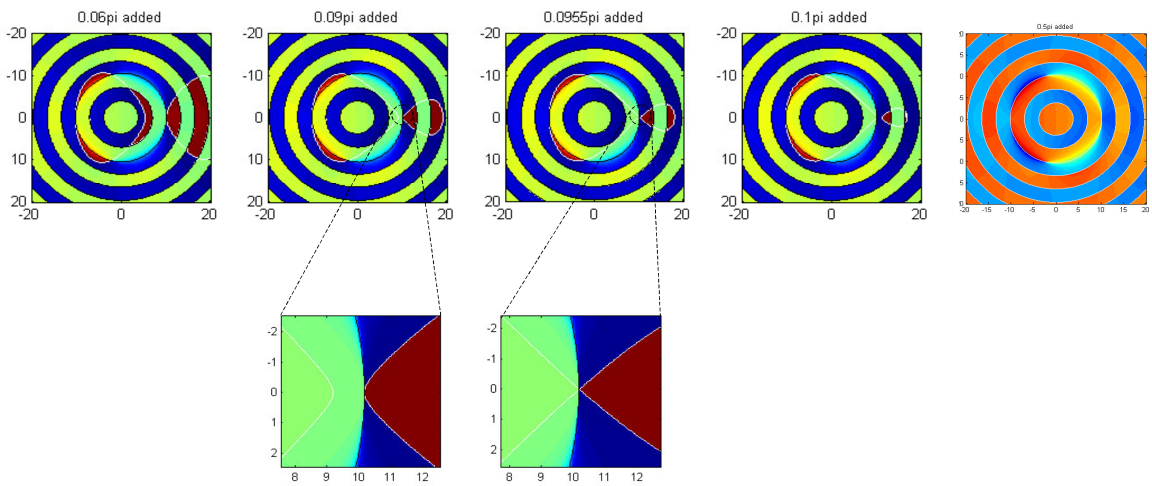



Fig. 9. Phasemaps for $a_3 = i\cos\left(\dfrac{\pi x}{2 j_3}\right); b_3 = i\sin\left(\dfrac{y}{50}\right)$ with increasing values of added phase.
Imaginary zero contours are overlaid in white, real zero contours in black.
Zoomed regions show branch point switch from one imaginary zero contour to the other.

An attempt was made to remove the branch cuts from this phasemap and to reconstruct the original object. All the regions where both the phase and the imaginary values are greater than zero (ie all the red/brown) regions of Fig. 8.(b)) had $2\pi$ subtracted from the phase, effectively unwrapping the phase discontinuity. The inverse Fourier transform of the wavefront was then taken and compared to that of the unmodified function. The values of the resulting objects are identical to within machine error. The inverse Fourier transform of the situation with $0.5\pi$ added phase was also considered. Again, this is identical to that of the original function to within machine error.

## 5 CONCLUSIONS

In band-limited complex functions of two complex variables it appears that zeros can take the form of isolated points or can form closed contours in the real plane. The asymptotic behaviour of such functions is expected to consist of concentric closed contours of zeros. Since it appears that branch cuts do not cross such lines, it follows that the branch points formed by isolated zeros either connect pairwise to each other or can connect to a line of zeros.

In terms of the effect of branch cuts on adaptive optics it is interesting to note that that we can alter the position of the branch cuts. This is an indication of the fact that they are not the source of reconstruction problems. Rather, the phase spiral around the branch point is the issue and removal of the branch cut affects this spiral. What we need to achieve in phase reconstruction algorithms is an assessment of the significance of the phase spiral.

Note: In the final stages of writing this paper we found a paper by Walford et al [7], which provides some elegant experimental data concerning the behaviour of branch points close to focus. These experiments relate closely to the modeling here, but we have had insufficient time to compare their published results to those presented here.

## 6 REFERENCES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